Extra credit problem:

Theorem

There exists a pair $\underset{\sim}{C}, \underset{\sim}{D} \ni P_{\underset{\sim}{C}}, P_{\underset{\sim}{D}}$ are r.e. inseperable but not effectively r.e. inseperable

Proof

Let A be an immune set whose complement is not r.e. By [Rogers Thm 8-II] there is such a set.

Let $\underset{\sim}{C} = \{\lambda y[n] \mid n \in \overline{A}\}$

Let $\underset{\sim}{D} = \{\lambda y[n] \mid n \in A\}$

1) $P_{\underset{\sim}{C}}$ is r.e. inseperable from $P_{\underset{\sim}{D}}$.

By s-m-n and padding there is a strictly increasing recursive function $f \ni (\forall x)[\varphi_{f(x)} = \lambda y[x]]$.

S bwoc $P_{\underset{\sim}{C}}$ is r.e. seperable from $P_{\underset{\sim}{D}}$. Then $\exists z\,[P_{\underset{\sim}{C}} \subseteq W_z$ and $P_{\underset{\sim}{A}} \subseteq \overline{W_z}]$. But then

$B = \{x \mid (\exists y)[y \in W_z \text{ and } f(x) = y]\}$ is r.e.

Moreover $x \in \overline{A} \Rightarrow f(x) \in P_{\underset{\sim}{C}} \subseteq W_z \Rightarrow x \in B$, $x \in A \Rightarrow f(x) \in P_{\underset{\sim}{D}} \Rightarrow f(x) \notin W_z \Rightarrow x \notin B$. Hence $\overline{A} = B$ and $\overline{A}$ r.e.. $\Rightarrow\!\!\!\!\times$

2) $P_{\underset{\wedge}{C}}$ is not effectively inseparable from $P_{\underset{\sim}{D}}$.

δ bwoc ow. Then $\exists$ rec $h \ni$

$$(\forall x)\left[h(x) \in (P_{\underset{\wedge}{C}} - W_x) \text{ or } h(x) \in W_x \cap P_{\underset{\sim}{D}}\right].$$

By the parametric recursion theorem $\exists$ recursive e such that

$$\varphi_{e(z)}(y) = \begin{cases} 0 & \text{if } [y = h(e(z)) \text{ and } \varphi_{h(e(z))}(0)\downarrow \notin D_z] \\ \uparrow & \text{otherwise} \end{cases}$$

Since $(\forall x)\left[h(x) \in P_{\underset{\wedge}{C}} \text{ xor } h(x) \in P_{\underset{\sim}{D}}\right]$; $(\forall z)\left[\varphi_{h(e(z))}(0)\downarrow \text{ and } [\varphi_{h(e(z))} \in \underset{\wedge}{C} \text{ xor } \varphi_{h(e(z))} \in \underset{\sim}{D}]\right]$.

$D_z \subseteq A \Rightarrow h(e(z)) \in W_{e(z)}$. For, $D_z \subseteq A \wedge h(e(z)) \notin W_{e(z)} \Rightarrow \varphi_{h(e(z))}(0) \in D_z \Rightarrow \exists u \in \underset{\sim}{D} \ni \varphi_{h(e(z))} = u \Rightarrow h(e(z)) \in P_{\underset{\sim}{D}}$. But $h(e(z)) \notin W_{e(z)} \Rightarrow h(e(z)) \in P_{\underset{\wedge}{C}}$. $\Rightarrow\Leftarrow$.

$\therefore D_z \subseteq A \Rightarrow h(e(z)) \in W_{e(z)} \wedge h(e(z)) \in P_{\underset{\sim}{D}}$. But $\varphi_{h(e(z))}(0) \notin D_z$.

$\therefore D_z \subseteq A \Rightarrow \varphi_{h(e(z))}(0) \in (A - D_z)$.

$\longrightarrow$

For all finite sets A, Let cA denote the canonical index of A.

Define g as follows:

$$g(0) = \varphi_{h(e(c\emptyset))}(0)$$

$$g(x+1) = \varphi_{h(e(c\{g(0), g(1), \ldots, g(x)\}))}(0)$$

g is clearly recursive since $(\forall z)[\varphi_{h(e(z))}(0)\downarrow]$.

Since $\emptyset \subseteq A$, $g(0) \in (A - \emptyset) = A$. $g(1) \in (A - \{g(0)\})$. $g(2) \in (A - \{g(0), g(1)\})$. By induction, $(\forall x)[(\forall y)[y \leq x \Rightarrow g(y) \in A]$ and $g(x+1) \in (A - \{g(0), g(1), \ldots, g(x)\})]$.

Hence $g(0), g(1), \ldots$ is a non-repeating enumeration of an infinite subset of A.
Hence A has an infinite r.e. subset but A is immune. Contradiction. $\square$

Good

believe!

Theorem: Suppose that $\underset{\sim}{C}, \underset{\sim}{D}$ are disjoint classes of partial recursive functions, where $\underset{\sim}{C}$ is a singleton $\{\psi\}$. Then $P_{\underset{\sim}{C}}$ is effectively r.e. inseparable from $P_{\underset{\sim}{D}}$ if and only if:

$$(\exists \text{ recursive } f)(\forall z)[D_z \subseteq \delta\psi \Rightarrow \psi\lceil_{D_z} \subseteq \varphi_{f(z)} \in \underset{\sim}{D}],$$

where the notation $\delta\psi$ means the domain of ψ, and $\psi\lceil_{D_z}$ means ψ with its domain restricted to D_z, i.e., $\lambda y\,[\text{if } y \in D_z \text{ then } \psi(y) \text{ else } \uparrow]$.

Proof:

$\Leftarrow$: Suppose there is a recursive function f such that $(\forall z)[D_z \subseteq \delta\psi \Rightarrow \psi\lceil_{D_z} \subseteq \varphi_{f(z)} \in \underset{\sim}{D}]$. By the parametric recursion theorem, there exists a recursive function e such that for all x, y:

$\varphi_{e(x)}(y)$ is computed by simultaneously enumerating $\delta\psi$ and W_x (alternately executing one step from each enumeration), to yield:

$\psi(y)$, if y appears in $\delta\psi$ before $e(x)$ appears in W_x;

$\varphi_{f(z)}(y)$, if $e(x)$ appears in W_x, and y does not appear in $\delta\psi$ before $e(x)$ appears in W_x, and D_z is the set of integers that appear in $\delta\psi$ before $e(x)$ appears in W_x;

$\uparrow$, if neither y appears in $\delta\psi$, nor $e(x)$ appears in W_x

Let x be any integer. Then there are two cases to consider:

case (i): $e(x) \in W_x$. In this case, let z be an integer such that D_z is the set of integers that appear in $\delta\psi$ before $e(x)$ appears in W_x. Then for all y, $\varphi_{e(z)}(y)$ is $\psi(y)$ if $y \in D_z$, and $\varphi_{f(z)}(y)$ if $y \notin D_z$. But $D_z \subseteq \delta\psi$ and therefore $\psi \restriction_{D_z} \subseteq \varphi_{f(z)} \in \underset{\sim}{D}$ by the hypothesis. Therefore, for all y, $\varphi_{e(x)}(y) = \varphi_{f(z)}(y)$ and $\varphi_{e(x)} \in \underset{\sim}{D}$.

case (ii): $e(x) \notin W_x$. In this case, for all y, $\varphi_{e(x)}(y) = \psi(y)$. Therefore, $\varphi_{e(x)} \in \underset{\sim}{C}$.

Thus, for any x, either $e(x) \in W_x \cap P_{\underset{\sim}{D}}$ or $e(x) \in P_{\underset{\sim}{C}} - W_x$. Thus, $P_{\underset{\sim}{C}}$ is effectively r.e. inseparable from $P_{\underset{\sim}{D}}$ as witnessed by the recursive function e. Q.E.D $\Leftarrow$.

$\Rightarrow$: Suppose $P_{\underset{\sim}{C}}$ and $P_{\underset{\sim}{D}}$ are effectively r.e. inseparable, where $\underset{\sim}{C}$ is the singleton $\{\psi\}$.

Then there must be a recursive function h such that $(\forall x)[h(x) \in W_x \cap P_{\underset{\sim}{D}}$ or $h(x) \in P_{\underset{\sim}{C}} - W_x]$.

By the parametric recursion theorem, there exists a recursive function e such that for all z, y:

$$\varphi_{e(z)}(y) = \begin{cases} 0, & \text{if } y = h(e(z)) \text{ and } (\forall u \in D_z)[\psi(u)\downarrow] \\ & \text{and } (\forall u \in D_z)[\varphi_{h(e(z))}(u)\downarrow = \psi(u)]; \\ \uparrow, & \text{otherwise.} \end{cases}$$

Let z be any integer such that $(\forall u \in D_z)[\psi(u)\downarrow]$, i.e. $D_z \subseteq \delta\psi$.

$h(e(z)) \in W_{e(z)}$. For if $h(e(z)) \notin W_{e(z)}$, i.e., $\varphi_{e(z)}(h(e(z)))\uparrow$, then not $(\forall u \in D_z)[\varphi_{h(e(z))}(u)\downarrow = \psi(u)]$, i.e., $\varphi_{h(e(z))} \neq \psi$. But $h(e(z)) \notin W_{e(z)} \Rightarrow$ $h(e(z)) \in P_{\underset{\sim}{C}} \Rightarrow \varphi_{h(e(z))} = \psi$. $\Rightarrow\Leftarrow$.

Since $h(e(z)) \in W_{e(z)}$, $\varphi_{e(z)}(h(e(z)))\downarrow$. Therefore,

$(\forall u \in D_z)[\varphi_{h(e(z))}(u)\downarrow = \psi(u)]$, i.e., $\psi \upharpoonright_{D_z} \subseteq \varphi_{h(e(z))}$.

Moreover, $h(e(z)) \in W_{e(z)} \Rightarrow h(e(z)) \in P_{\underset{\sim}{D}}$; so

$\varphi_{h(e(z))} \in \underset{\sim}{D}$.

Therefore, $(\forall z)[D_z \subseteq \delta\psi \Rightarrow \psi\upharpoonright_{D_z} \subseteq \varphi_{h(e(z))} \in \underset{\sim}{D}]$,

and $\lambda z[h(e(z))]$ is the required recursive function. Q.E.D. $\Rightarrow$.

Q.E.D. proof.